# Assignment 4.

This homework is due *Thursday*, September 27.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper and *credit your collaborators*. Your solutions should contain full proofs. Bare answers will not earn you much. Extra problems (if there are any) are due December 7.

## 1. Exercises

(1) (1.4.47) Let $E$ be a closed set of real numbers and $f : E \to \mathbb{R}$ be continuous. Show that there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g|_E = f$. (*Hint:* Take $g$ to be linear on each of the intervals of which $\mathbb{R} \setminus E$ is composed.)
Note: whenever a function is said to be continuous, without specifying the set on which it is continuous, it is implied that the function is continuous on its domain.

(2) ($\sim$1.4.49) Let $f, g : E \to \mathbb{R}$ be continuous.
  (a) Let $\max\{f, g\} : E \to \mathbb{R}$ be the function defined by $\max\{f, g\}(x) = \max\{f(x), g(x)\}$, $x \in E$. Show that that $\max\{f, g\}$ is continuous.
  (b) Show that $|f|$ is continuous.
(*Hint:* Show that $\max\{a, b\} = \frac{a+b+|a-b|}{2}$ and $|a| = \max\{a, -a\}$. Conclude that it is enough to prove one of the above statements.)

(3) (1.4.51) (Approximation of continuous functions by piecewise linear ones) A continuous function $\varphi$ on $[a, b]$ is called *piecewise linear* provided there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ of $[a, b]$ for which $\varphi$ is linear on each interval $[x_i, x_{i+1}]$.
Let $f$ be continuous on $[a, b]$ and $\varepsilon$ a positive number. Show that there is a piecewise linear function $\varphi$ on $[a, b]$ with $|f(x) - \varphi(x)| < \varepsilon$ for all $x \in [a, b]$. (*Hint:* Use uniform continuity.)

(4) (Brouwer theorem for a segment) Let $f : [0, 1] \to \mathbb{R}$ be continuous and $f([0, 1]) \subseteq [0, 1]$ (i.e. all values of $f$ are contained in $[0, 1]$). Then there is a point $x \in [0, 1]$ such that $f(x) = x$. (*Hint:* Apply intermediate value theorem to a suitable function.)

(5) (1.4.52) Show that a nonempty subset $E$ of $\mathbb{R}$ is closed and bounded if and only if every continuous real-valued function on $E$ takes a maximum value.

(6) (1.4.58) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Prove that the inverse image w.r.t. $f$ of every closed set is closed, and of every Borel set is Borel. (*Hint:* Show that $f^{-1}$ respects set-theoretic operations.)

— see next page —

## 2. Extra exercises

(7) Can 2a be generalized to arbitrary family of functions?

(a) Let $\mathcal{F}$ be a (possibly infinite) family of continuous functions $E \to \mathbb{R}$. Assume additionally that this family is uniformly bounded above: there is $M \in \mathbb{R}$ s.t. for all $f \in \mathcal{F}$, for all $x \in E$, $f(x) < M$. Is it true that the function $\sup \mathcal{F}$ defined by $\sup \mathcal{F}(x) = \sup\{f(x) \mid f \in \mathcal{F}\}$ is also continuous?

(b) What if additionally this family is *equicontinuous* on E, i.e. for any $\varepsilon > 0$, there is $\delta > 0$ such that for all $x, y \in E$ and all $f \in \mathcal{F}$, if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$?

(8) Prove that the set of all continuous functions $\mathbb{R} \to \mathbb{R}$ (or $[0,1] \to \mathbb{R}$, whichever you like better) is equipotent to $\mathbb{R}$ (i.e. that there is a bijection between this set and $\mathbb{R}$). You can take Problem 8 of Assignment 2 for granted.